EXAMPLES OF PSEUDO-MINIMAL TRIANGULATIONS

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Abstract. Examples of pseudo-minimal triangulations on various surfaces are given.

1. Introduction

A *triangulation* of a closed surface is a simple graph embedded in the surface so that each face is a triangle and so that any two faces share at most one edge. Two triangulations T and T' of a surface are *equivalent* if there is a isomorphism h with $h(T) = T'$. That is, if a, b, and c are vertices of T then ab is an edge of T if and only if $h(a)h(b)$ is an edge of T' and a face of T is bounded by the cycle abc if and only if a face of T' is bounded by the cycle $h(a)h(b)h(c)$. Two edges of a triangulation are *equivalent* if there is an automorphism of the triangulation which maps one edge into the other.

Let ac be an edge in a triangulation T and abc and acd be the two faces which have ac as a common edge. The *contraction* of ac is obtained by deleting ac, identifying vertices a and c, removing one of the multiple edges ab or cb, and removing one of the multiple edges ad or cd. The edge ac of a triangulation T is *contractible* if the contraction of ac yields another triangulation of the surface in which T is embedded. If the edge ac is contained in a 3-cycle other than the two which bound the faces which share it then its contraction would produce multiple edges. So, for a triangulation T, not K_4 embedded in the sphere, an edge of T is not contractible if and only if that edge is contained in at least three 3-cycles. A triangulation is said to be *contractible* if it has contractible edges. A triangulation is said to be *irreducible* if it has no contractible edge.

Let ac be an edge in a triangulation T and abc and acd be the two faces which have ac as a common edge. The *diagonal flip* of ac is obtained by deleting ac, adding edge bd, deleting the faces abc and acd, and adding the faces abd and bcd. An edge ac of a triangulation T is *flippable* if the diagonal flip of ac yields another triangulation of the surface in which T is embedded. So ac is flippable if bd is not already an edge. Two triangulations are *equivalent under diagonal flips* if one is equivalent to a triangulation obtained from the other by a sequence of diagonal flips.

The number of vertices of an irreducible triangulation can not be reduced by edge contraction. Negami [11] defines a type of triangulation for which the number of vertices can not be reduced by a combination of diagonal flips and edge contractions. An irreducible triangulation is said to be *pseudo-minimal* if it is not equivalent under diagonal flips to a contractible triangulation.

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Negami [13, 14] defines two types of triangulations which can easily be recognized as pseudo-minimal. A triangulation is said to be *frozen* if it has no flippable edges. An edge of a triangulation is said to be *self-flippable* if when it is flipped an equivalent triangulation is produced. A triangulation is said to be *isolated* if it is equivalent under diagonal flips only to itself. All the flippable edges of an isolated triangulation are self-flippable.

A triangulation is said to be *minimal* if there are no triangulations of the same surface with fewer vertices. It is clear that such a triangulation is also pseudo-minimal. The number of vertices in a minimal triangulation for nonorientable surfaces was determined by Ringel [16] and for orientable surfaces by Jungerman and Ringel [5]. It is given for all surfaces except N_2, N_3, and S_2 by the formula:

$$V_{min}(S) = \left\lceil \frac{7 + \sqrt{49 - 24\chi(S)}}{2} \right\rceil$$

For the three exceptions the value is one more than the value given by the formula: $V_{min}(N_2) = 8$, $V_{min}(N_3) = 9$, and $V_{min}(S_2) = 10$.

A triangulation is said to be *complete* if the embedded graph is complete. By the Euler formula K_n can be embedded as a triangulation in a surface S only if $n = (7 + \sqrt{49 - 24\chi(S)})/2$. So any complete triangulation must be minimal. According to the "Map Color Theorem" [17] complete triangulations of K_n exist for orientable surfaces if and only if $n \equiv 0$, 3, 4, or 7 mod 12 and $n \geq 4$ and complete triangulations of K_n exist for nonorientable if and only if $n \equiv 0$ or 1 mod 3 and $n \geq 6$ and $n \neq 7$.

For a fixed surface we can take a topographic approach to the study of the triangulations of that surface as suggested by Negami [12]. In such a view a triangulation, T, is at the same elevation and next to another, T', if T' is obtained from T by a diagonal flip. Also, a triangulation, T, is above a triangulation, T' if T' is obtained from T by an edge contraction. The topographic "surface" in such a view consists of the irreducible triangulations. Pseudo-minimal triangulations form the bottoms of the valleys. We will call these "lakes". A *lake* is the set of all pseudo-minimal triangulations of a surface which are equivalent under diagonal flips to some fixed pseudo-minimal triangulation. The *elevation* of a lake is the number of vertices in any pseudo-minimal triangulation in the lake. Let $L_{max}(S)$ be the maximum elevation of any lake of S which is the maximum number of vertices in any pseudo-minimal triangulation of S.

We will order the lakes of S with elevation nv arbitrarily and designate the i-th lake with elevation nv by $L(S, nv, i)$. We will order the pseudo-minimal triangulations in $L(S, nv, i)$ arbitrarily and designate the j-th pseudo-minimal triangulation in $L(S, nv, i)$ by $P(S, nv, i, j)$.

Let $N(S)$ be the minimum value such that two triangulations T and T' are equivalent under diagonal flips if the number of vertices in T and the number of vertices in T' are equal and at least $N(S)$. Negami [11] has shown that such a finite value exists for any S.

Theorem 1. *A surface S has exactly one lake if and only if*

$$N(S) = L_{max}(S) = V_{min}(S).$$

Proof: If $N(S) = L_{max}(S) = V_{min}(S)$ then all pseudo-minimal triangulations have $N(S)$ vertices and are equivalent under diagonal flips. Thus there is only one lake.

Negami [11] has shown that two triangulations of a closed surface with the same number of vertices are equivalent under diagonal flips if they can be transformed into a common triangulation by diagonal flips and contraction of edges. If a surface has only one lake then every triangulation of that surface can be transformed into any triangulation in that lake by diagonal flips and contraction of edges.

Theorem 2. *If a surface S has more than one lake then*

$$N(S) \geq L_{max}(S) + 1 \geq V_{min}(S) + 1.$$

Proof: Suppose a surface, S, has more than one lake. We will show that there are two triangulations of S with $L_{max}(S)$ vertices which are not equivalent under diagonal flips and so $N(S) \geq L_{max}(S) + 1$. There is pseudo-minimal triangulation T_1 in a lake of elevation $L_{max}(S)$ and another triangulation, T_2, in some other lake. The number of vertices in T_2 is no more than the number of vertices in T_1. If T_1 and T_2 have the same number of vertices then they are not equivalent under diagonal flips since they are in different lakes. If T_2 has fewer vertices than T_1 then there is a triangulation T_2^+ which has the same number of vertices as T_1 and which can be transformed into T_2 by contracting edges. Since T_1 is pseudo-minimal T_1 is not equivalent under diagonal flips to T_2^+ which is contractible.

2. Finding pseudo-minimal triangulations

The irreducible triangulations of S_0, S_1, S_2, N_1, N_2, N_3, and N_4 have been determined [18] [1] [7] [8] [20] [19] have been determined. The complete sets of pseudo-minimal triangulations were determined for these surfaces using the complete lists of irreducible triangulations.

Lutz [9] generated for all surfaces all the triangulations with 10 or fewer vertices. Sulanke and Lutz [21] generated for all surfaces all the triangulations with 12 or fewer vertices. Using these results we were able to check that the table is complete for pseudo-minimal triangulations up to 12 vertices.

Surface	V_{min}	t	Number of Pseudo-minimal
S_0	4	0	$1 = (1)_4$
N_1	6	0	$1 = (1)_6$
S_1	7	0	$1 = (1)_7$
N_2	8	4	$6 = (6)_8$
N_3	9	6	$133 = (133)_9$
S_2	10	9	$865 = (865)_{10}$
N_4	9	3	$37 = (32 + 3 + 2)_9$
N_5	9	0	$2 = (2 * 1)_9$
S_3	10	3	$20 = (13 + 5 + 2 * 1)_{10}$
N_6	10	3	$1022 = (363 + 297 + 253 + 24 + 12 + 2 * 11 + 8 + 6 + 2 * 5 + 2 * 4 + 3 * 3 + 4 * 2 + 2 * 1)_{10}$
N_7	10	0	$34 = 14 + 20 = (14 * 1)_{10} + (8 + 2 * 4 + 3 + 1)_{11}$
S_4	11	4	$823 = 821 + 2 = (786 + 9 + 2 * 4 + 5 * 3 + 3 * 1)_{11} + (2 * 1)_{12}$
N_8	11	4	$295302 = 295291 + 11 = (290756 + .. + 257 * 1)_{11} + (4 + 3 + 4 * 1)_{12}$
N_9	11	1	$9864 = 5982 + 3882 = (211 + .. + 1336 * 1)_{11} + (2 * 48 + .. + 65 * 1)_{12}$

3. The examples

The figures of triangulations shown here consist of four parts. On the left of each figure is a polygon representing the triangulation. The polygons used for the projective plane, torus, and Klein bottle are those commonly used for these surfaces and are taken from the original references. For the other surfaces no attempt is made to fit any standard form. To see that the polygons represent the stated surface several checks must be made. Each boundary edge of the polygon must appear exactly twice. If both representations of an edge on the boundary have the same orientation going around the boundary then the surface is nonorientable. The order of the neighbors around a vertex must agree with the rotation which is given in the upper middle of each figure. The thicker lines represent flippable edges.

In the lower middle of each figure is a list of generators for the automorphism group of the triangulation. These generators can be used to check the claims concerning isomorphic edges, etc. If the automorphism group is trivial then no generators are shown. The generators were produced by nauty [10] by first replacing the faces with new vertices of degree 3.

On the right of each figure is the complement of the embedded graph. The triangulations which we are considering are dense as graphs so their complements are sparse. The complements are included for two reason. First, they are sometimes useful in checking that two triangulations are nonequivalent. Nonequivalence of the complements is usually easier to recognize than the nonequivalence of the triangulations as graphs. Nonequivalent graphs provide nonequivalent triangulations. Second, when flipping an edge the new edge that is added to the triangulation must come from the complement. It is usually quicker to check the complement for available additions than to check the triangulation for flippable edges. Two vertices of a triangulation are *opposite* each other across an edge not containing them if they are on faces which share that edge. An edge bd from the complement allows a flip if b and d are opposite each other in the triangulation across a flippable edge ac. b and d can be opposite each other across many edges or none at all. The number, if any,

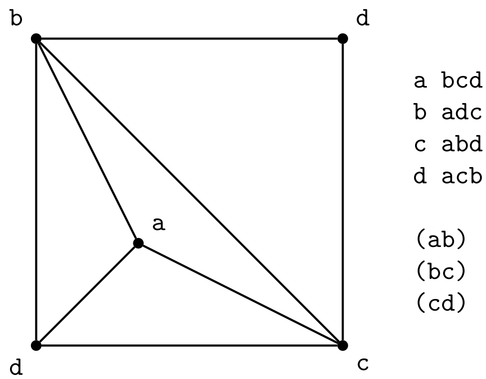


FIGURE 1. $P(S_0, 4, 1, 1) = K_4$

shown on an edge in the drawing of the complement is the number of edges in the triangulation which can be flipped and replaced by that edge. The total of these edge numbers must equal the number of flippable edges shown in the representation of the triangulation.

We now consider in turn several different surfaces and some of the properties of the pseudo-minimal triangulations of each surface.

4. S_0, THE SPHERE

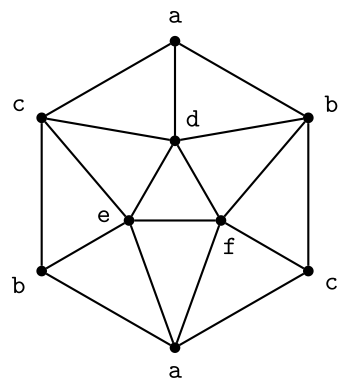
By Steinitz' Theorem [18] all triangulations of the sphere can be generated from the triangulation of K_4 by a sequence of vertex splittings. K_4 which is shown in Figure 1 is thus the only irreducible triangulation of the sphere. The sphere has just one lake which contains only K_4. $N(S_0) = L_{max}(S_0) = V_{min}(S_0) = 4$.

5. N_1, THE PROJECTIVE PLANE

Barnette [1] showed that all triangulations of the projective plane can be generated from the triangulations shown in Figs. 2 and 3 by a sequence of vertex splittings. These two triangulations are thus the only irreducible triangulations of the projective plane. The triangulation in Figure 3 is not pseudo-minimal since if we flip dg then ag becomes a contractible edge. Thus $P(N_1, 6, 1, 1)$ in Figure 2 is the only pseudo-minimal triangulation of the projective plane. $P(N_1, 6, 1, 1)$ is also complete. The projective plane has just one lake which contains only the triangulation $P(N_1, 6, 1, 1)$. $N(N_1) = L_{max}(N_1) = V_{min}(N_1) = 6$.

6. S_1, THE TORUS

Dewdney [3] showed that every triangulation of the torus is equivalent under diagonal flips to a triangulation which can then be reduced by a sequence of edge contractions to the triangulation $P(S_1, 7, 1, 1)$ in Figure 4. Thus $P(S_1, 7, 1, 1)$ is the only pseudo-minimal triangulation of the torus. $P(S_1, 7, 1, 1)$ is also complete. The torus has just one lake which contains only $P(S_1, 7, 1, 1)$. $N(S_1) = L_{max}(S_1) = V_{min}(S_1) = 7$.



a bdcfe
b adfce
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d abfec
e abcdf
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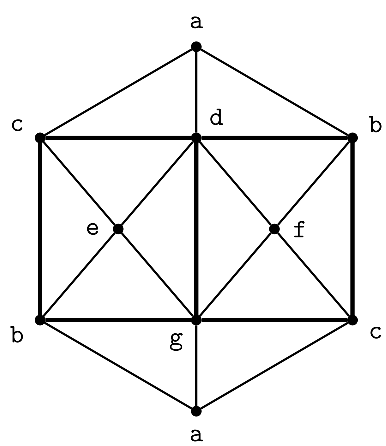
FIGURE 2. $P(N_1, 6, 1, 1) = K_6$



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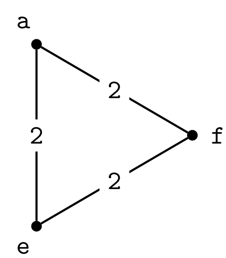

FIGURE 3. The irreducible triangulation of the projective plane which is not pseudo-minimal

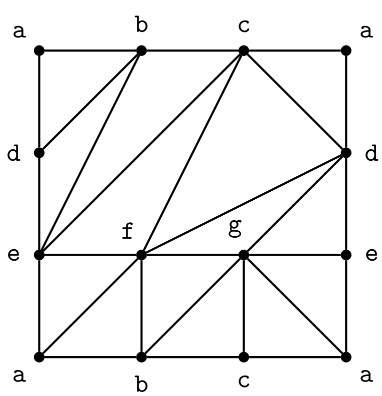

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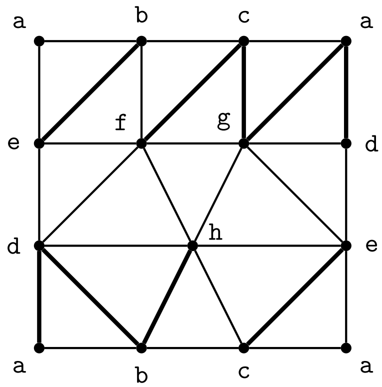
FIGURE 4. $P(S_1, 7, 1, 1) = K_7$



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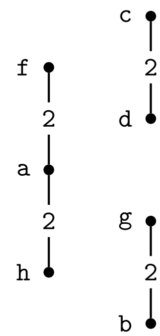
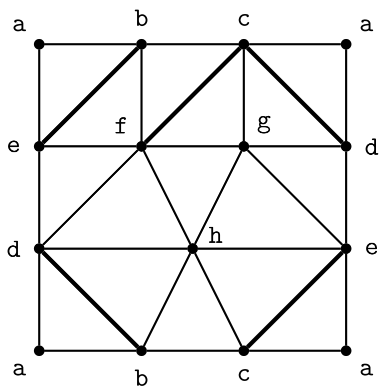


FIGURE 5. $P(N_2, 8, 1, 4) = \text{Kh4}$



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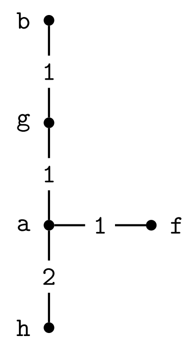


FIGURE 6. $P(N_2, 8, 1, 6) = \text{Kh6}$

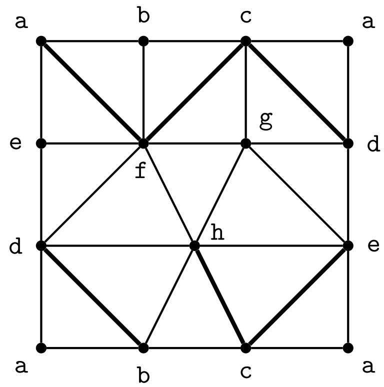
7. N_2, THE KLEIN BOTTLE

Negami and Watanabe [15] showed that every triangulation of the Klein bottle is equivalent under diagonal flips to a triangulation which can then be reduced by a sequence of edge contractions to the triangulation Kh6 in Figure 6. This means that Kh6 is pseudo-minimal and is in the only lake of the Klein bottle. Lawrencenko and Negami [8] determined all the pseudo-minimal triangulations in this lake which are shown in Figs. 5 through 10. Kh3 has been redrawn to more obviously show that it can be obtained from Kh1 with a single flip. $N(N_2) = L_{max}(N_2) = V_{min}(N_2) = 8$.

8. N_3

For N_3 there is only one lake with 133 pseudo-minimal triangulations. Recall that N_3 is one of the three exceptions to the formula for V_{min}. $V_{min}(N_3)$ is one larger than is required by the Euler formula.

Two of these triangulations are shown in Figures 11 and 12. All 133 are shown in Figure 13. Due to space limitation only the unlabeled polygons corresponding to the triangulations are shown. Each polygon in Figure 13 is labeled the same as the two shown in Figures 11 and 12. In Figure 13 if two polygons are adjacent and not separated by a dark line then one can be obtained from the other by a



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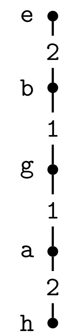
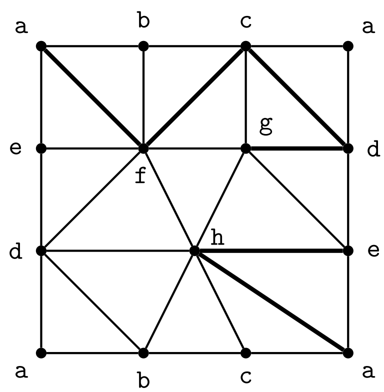


FIGURE 7. $P(N_2, 8, 1, 5) = \text{Kh5}$



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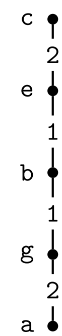
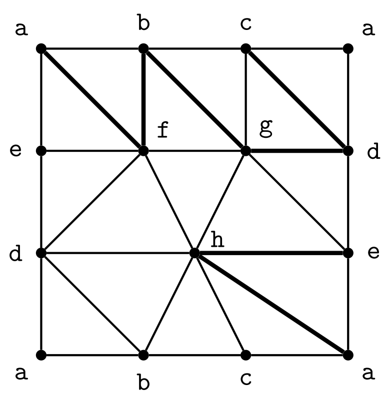


FIGURE 8. $P(N_2, 8, 1, 2) = \text{Kh2}$



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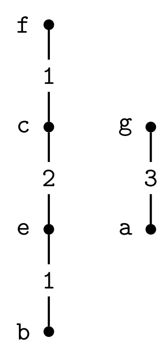
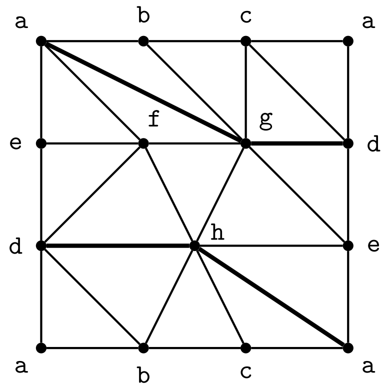


FIGURE 9. $P(N_2, 8, 1, 1) = \text{Kh1}$

single diagonal flip. Since the maze formed by the dark lines is connected we see that any two polygons connected by a sequence of diagonal flips. Thus the 133 triangulations are equivalent under diagonal flips. In order to check the claim that these 133 triangulations do form a lake it is necessary to check that each polygon represents a unique triangulation of N_3 and that there are no other triangulations



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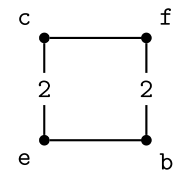
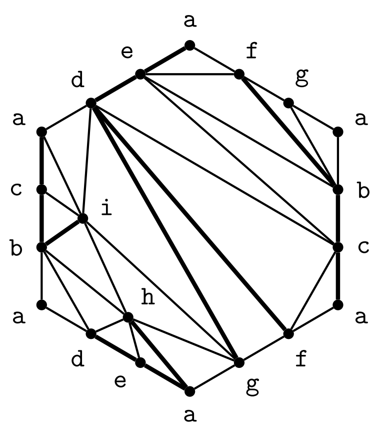


FIGURE 10. $P(N_2, 8, 1, 3) = \text{Kh3}$ (redrawn)



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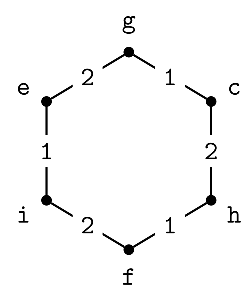
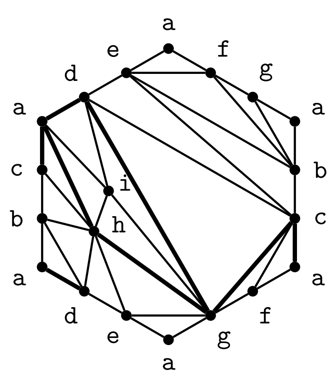


FIGURE 11. $P(N_3, 9, 1, 1)$



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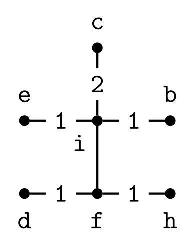


FIGURE 12. $P(N_3, 9, 1, 2)$

which are equivalent under diagonal flips to any of these. This was done when the triangulations were generated as described in Section 2.

These 133 triangulations are all of the pseudo-minimal triangulations of N_3. So $N(N_3) = L_{max}(N_3) = V_{min}(N_3) = 9$.

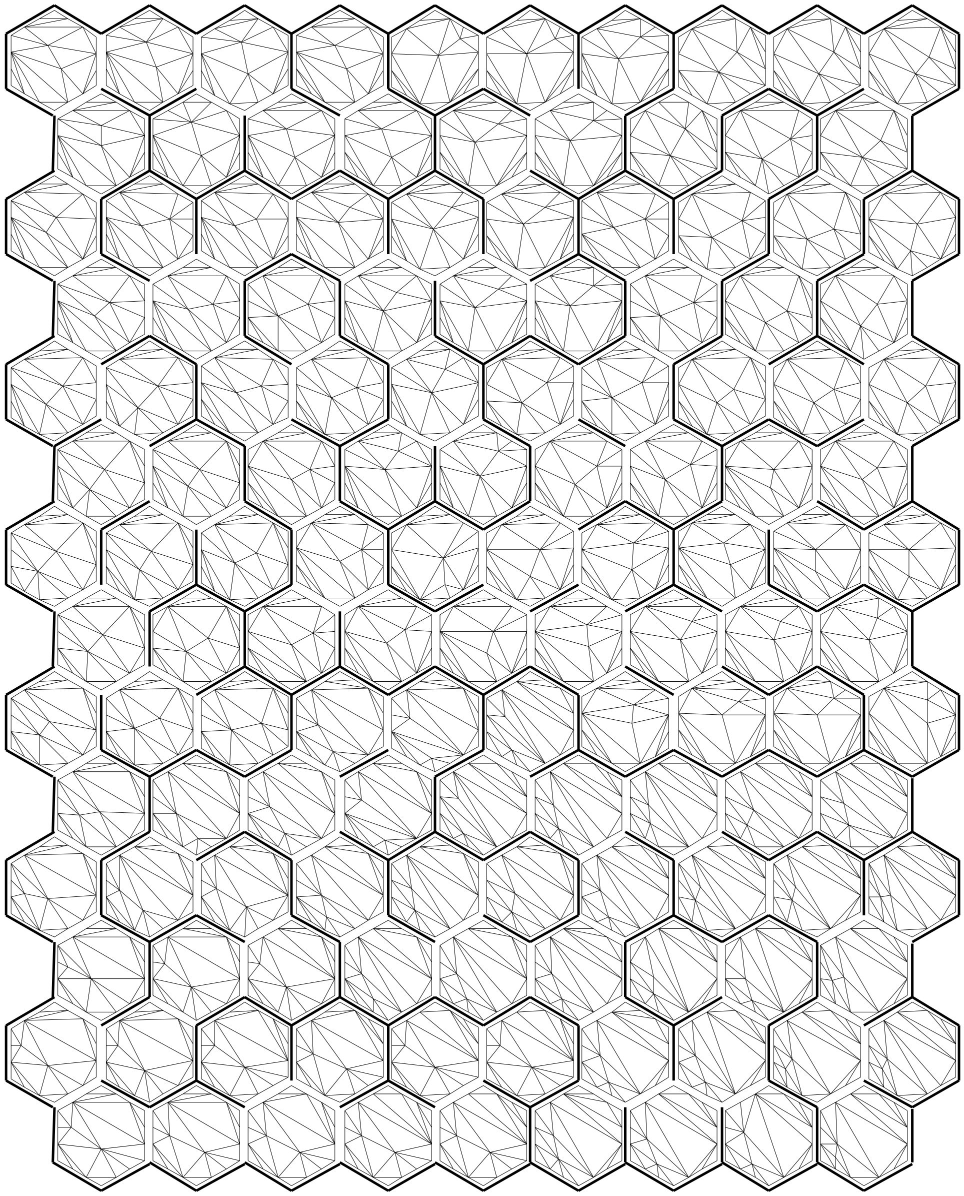
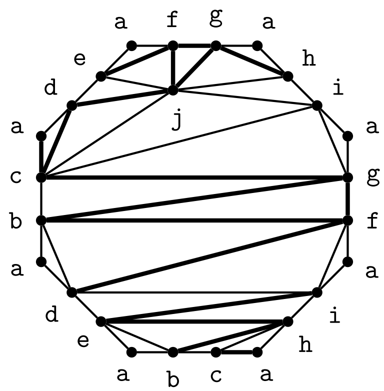


FIGURE 13. $L(N_3, 9, 1)$

9. S_2, THE DOUBLE TORUS

For S_2 there is only one lake with 865 pseudo-minimal triangulations. Again S_2 is one of the three exceptions to the formula for V_{min} as was shown in [4]. In [4] Huneke also provides an example of a pseudo-minimal triangulation of S_2 with 10 vertices. This triangulation is shown here in Figure 14. By flipping ef, cg, and bf we get Figure 15 and by flipping dj, ej, and df we get Figure 16. These three pseudo-minimal triangulations from $L(S_2, 10, 1)$ have been redrawn in



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(ce)(dh)(gf)(jb)(ai)
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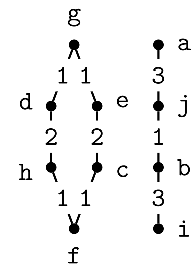
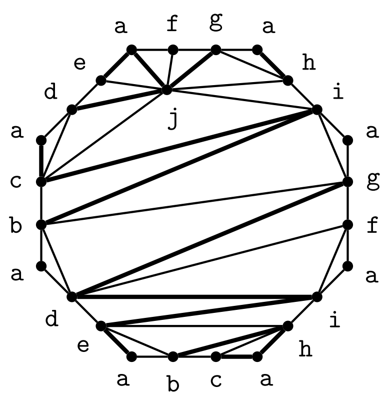


Figure 14. $P(S_2, 10, 1, 1)$ from Huneke



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(bh)(dj)(ia)
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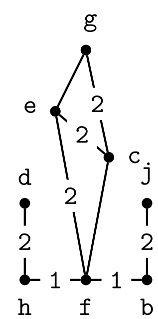
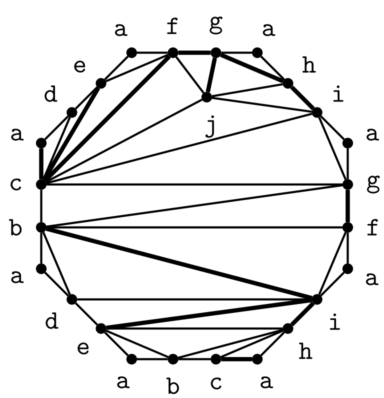


Figure 15. $P(S_2, 10, 1, 2)$



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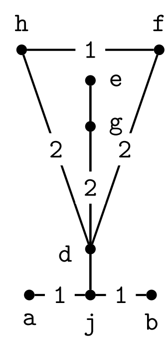
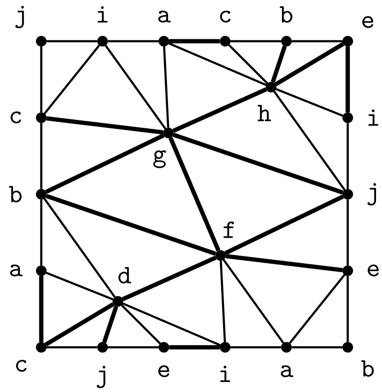


Figure 16. $P(S_2, 10, 1, 3)$

Figures 17 through 19 in an attempt to better show their automorphisms. The 865 triangulations of S_2 with 10 vertices are all of the pseudo-minimal triangulations of S_2. So $N(S_2) = L_{max}(S_2) = V_{min}(S_2) = 10$.



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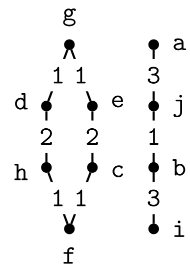
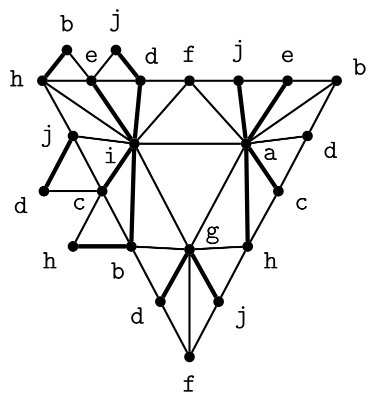


FIGURE 17. $P(S_2, 10, 1, 1)$ redrawn



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```

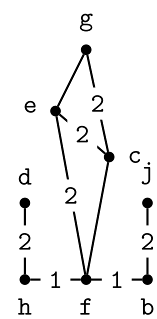
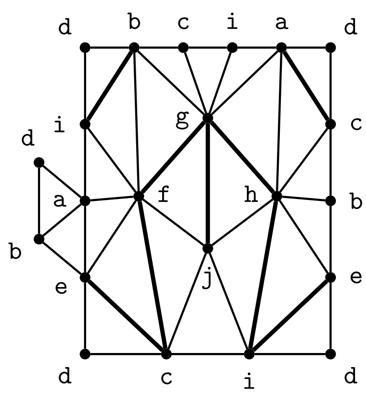


FIGURE 18. $P(S_2, 10, 1, 2)$ redrawn



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i agcjhedbf
j cfghi

(fh)(ab)(ic)
```

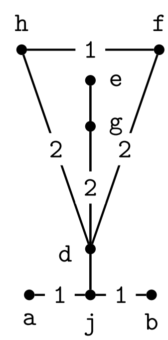
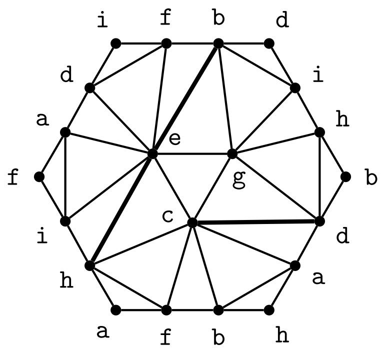


FIGURE 19. $P(S_2, 10, 1, 3)$ redrawn

10. N_4

Finally a surface with more than one lake. There are three lakes with 32, 3, and 2 triangulations. We will show here that there are at least three lakes. None of the



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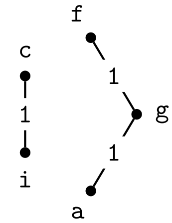
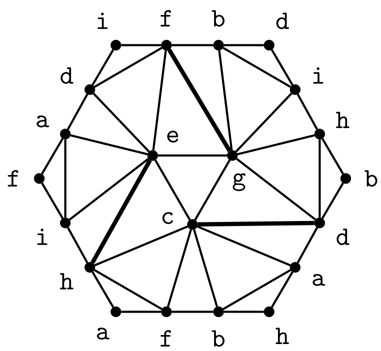


FIGURE 20. $P(N_4, 9, 3, 1)$



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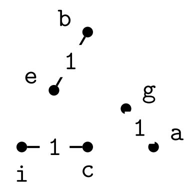


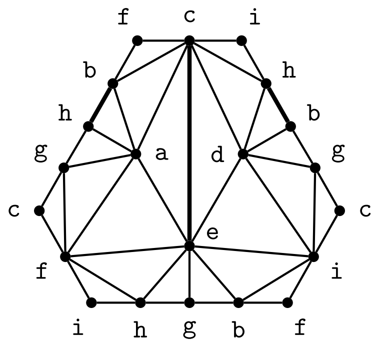
FIGURE 21. $P(N_4, 9, 3, 2)$

lakes have just one pseudo-minimal triangulation so there are no frozen or isolated triangulations.

Figures 20 and 21 show the two triangulations in $L(N_4, 9, 3)$. The edge fg of $P(N_4, 9, 3, 2)$ is equivalent to all three of the flippable edges of $P(N_4, 9, 3, 2)$. Flipping fg produces $P(N_4, 9, 3, 1)$. $P(N_4, 9, 3, 1)$ has three nonequivalent flippable edges. Two edges, eh and cd, are self-flippable while the third, be, produces $P(N_4, 9, 3, 2)$ when flipped. So $P(N_4, 9, 3, 1)$ and $P(N_4, 9, 3, 2)$ are equivalent under flipping to no other triangulations.

Figures 22, 23 and 24 show the three triangulations in $L(N_4, 9, 2)$. Consider $P(N_4, 9, 2, 3)$. The edge ad is equivalent to all three of the flippable edges. Flipping ad produces $P(N_4, 9, 2, 1)$. Now consider $P(N_4, 9, 2, 1)$. Flipping ce produces $P(N_4, 9, 2, 3)$. Flipping bh produces $P(N_4, 9, 2, 2)$. This can be seen by examining the differences between the rotations of $P(N_4, 9, 2, 1)$ and $P(N_4, 9, 2, 2)$. Flipping bh removes h from the rotation of b and removes b from the rotation of h. The new edge ad in $P(N_4, 9, 2, 2)$ is obtained by adding d in the rotation of a between b and h and adding a in the rotation of d between b and h. Finally consider $P(N_4, 9, 2, 2)$. Flipping ad, which is equivalent to fi, produces $P(N_4, 9, 2, 1)$ again. Edge eg is self-flippable.

Figure 25 shows $P(N_4, 9, 1, 6)$ yet another pseudo-minimal triangulation of N_4. That it is unique can be seen by comparing its complement with those of the pseudo-minimal triangulations in $L(N_4, 9, 2)$ and $L(N_4, 9, 3)$. Since it is not in $L(N_4, 9, 2)$ and not in $L(N_4, 9, 3)$ there must be at least three lakes for N_4.

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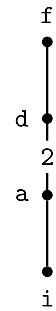
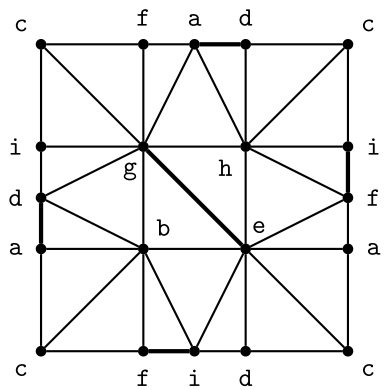


FIGURE 22. $P(N_4, 9, 2, 1)$

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(bh)(da)(if)
```

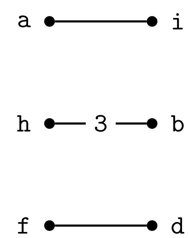
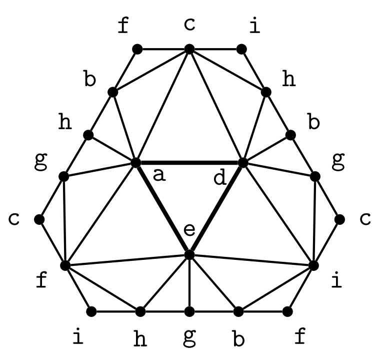


FIGURE 23. $P(N_4, 9, 2, 2)$

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h ageficdb
i bedgchf

(fi)(ad)(bh)
(cf)(de)(bg)
```

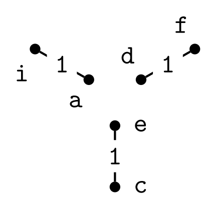


FIGURE 24. $P(N_4, 9, 2, 3)$

In Figure 26 is shown a sequence of operations for obtaining $P(N_4, 9, 1, 6)$ from $P(N_4, 9, 2, 1)$. Since these two pseudo-minimal triangulations are in different lakes a simple sequence of flips is not enough. Starting in the lower left of Figure 26 we move up (increase the number of vertices by one) from $P(N_4, 9, 2, 1)$ by splitting the vertex i to create a new vertex j. The triangulation in the upper left of Figure 26

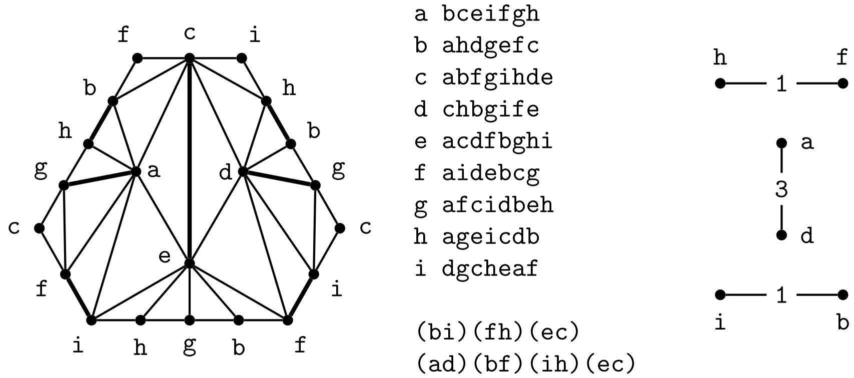


FIGURE 25. $P(N_4, 9, 1, 6)$

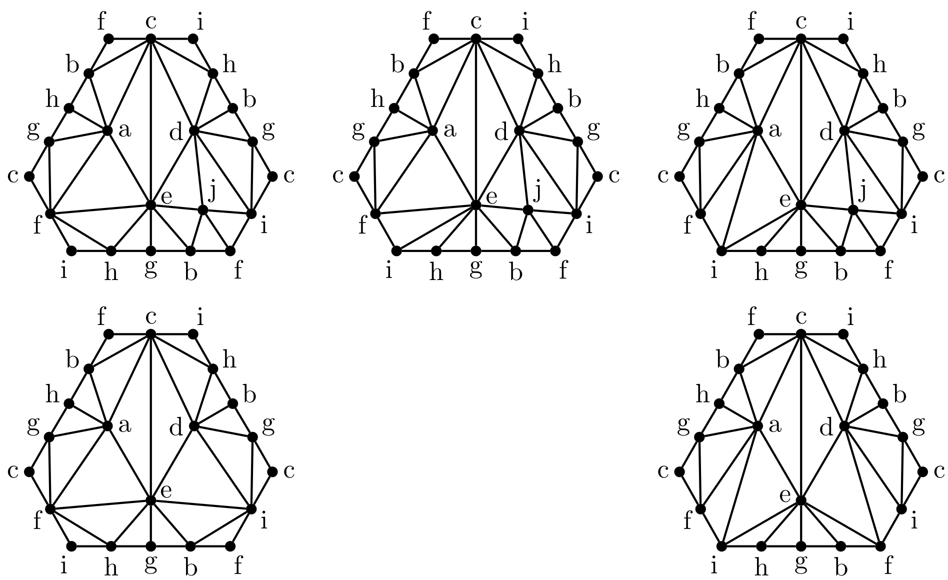
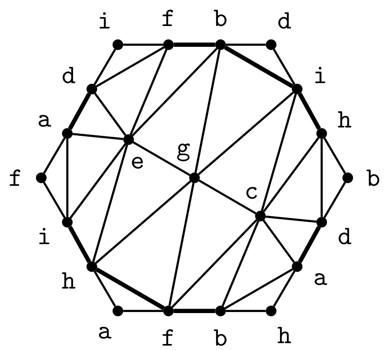


FIGURE 26. Portage from $P(N_4, 9, 2, 1)$ to $P(N_4, 9, 1, 6)$

is not irreducible since the edge ij can be contracted to obtain $P(N_4, 9, 2, 1)$. If the edge fh is flipped we obtain the irreducible triangulation in the upper center of Figure 26 maintaining the same number of vertices. If the edge fe is then flipped we obtain the triangulation in the upper right of Figure 26 also with the same number of vertices. This triangulation is not irreducible since we can contract the edge fj and obtain $P(N_4, 9, 1, 6)$ in the lower right of Figure 26. This contraction again reduces the number of vertices to the elevation of the lakes. We call this sequence of triangulations a portage from $L(N_4, 9, 2)$ to $L(N_4, 9, 1)$.

A *portage* from lake L_1 to lake L_2 of *length* n is a sequence of triangulations $(T_1, T_2, ..., T_n)$ where an edge of T_1 can be contracted to obtain a pseudo-minimal triangulation of L_1, an edge of T_n can be contracted to obtain a pseudo-minimal triangulation of L_2, T_i is irreducible for $i = 2, .., n-1$, and T_i is obtained from T_{i-1} by flipping an edge for $i = 2, .., n$. We allow $n = 1$ or $n = 2$. However, no such




```
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b ahdigefc
c abfgihd
d chbifea
e adfbghi
f aidebcgh
g fcibeh
h afgeicdb
i dbgcheaf

(ce)(bi)(fh)
(ce)(ad)(bf)(ih)
```

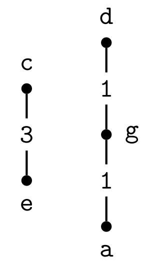


FIGURE 27. $P(N_4, 9, 1, 15)$

short portages were found for any surface. T_1 and T_n are the *ends* of the portage. The *portage graph* of a surface has the lakes of the surface as vertices. A pair of lakes is an edge in the portage graph if there is a portage between the lakes. The weight of an edge in the portage graph is the minimum length of portages between the two lakes which are the edge's ends.

Theorem 3. *If a surface S has more than one lake and the portage graph of S is connected then*

$$N(S) = L_{max}(S) + 1.$$

Proof: Suppose that a surface S has more than one lake and that the portage graph of S is connected. From Theorem 2 we have $N(S) \geq L_{max}(S) + 1$. We show that $N(S) \leq L_{max}(S)+1$. All the lakes must have the elevation of $L_{max}(S)$ Suppose T is a triangulation with $L_{max}(S) + 1$ vertices. T is not pseudo-minimal so it is equivalent under diagonal flips to a triangulation T_a which is contractible to some pseudo-minimal triangulation in a lake L_a. T_a is equivalent under diagonal flips to all triangulations which are contractible to some pseudo-minimal triangulation in a lake L_a. In particular, T_a is equivalent under diagonal flips to the L_a end of all portages beginning at lake L_a. Since the two ends of any portage are equivalent under diagonal flips by definition T_a is also equivalent under diagonal flips to the non-L_a end of all portages beginning at lake L_a. Continuing in this way we see that T is equivalent under diagonal flips to every triangulation which is contractible to some pseudo-minimal triangulation. So any two triangulation with $L_{max}(S) + 1$ vertices are equivalent under diagonal flips.

Figure 27 shows a second pseudo-minimal triangulation in $L(N_4, 9, 1)$. It can be obtained from $P(N_4, 9, 1, 6)$ by flipping in any order the edges ag, ce, and dg. Figure 28 shows a portage from $L(N_4, 9, 3)$ to $L(N_4, 9, 1)$ of length 3 which begins over $P(N_4, 9, 3, 1)$ and ends over $P(N_4, 9, 1, 15)$. The operations are: split vertex c, flip edge dg, flip edge gh, and contract edge gj. The portages which we have shown are not the only ones. However, there are no shorter ones between the respective lakes. The shortest portage from $L(N_4, 9, 2)$ to $L(N_4, 9, 3)$ has length 8.

The minimal spanning tree of the portage graph of N_4 is the path, P_3 where both edges have weight 3. These 37 triangulations are all of the pseudo-minimal triangulations of N_4. So $N(N_4) = L_{max}(N_4) + 1 = V_{min}(N_4) + 1 = 10$.

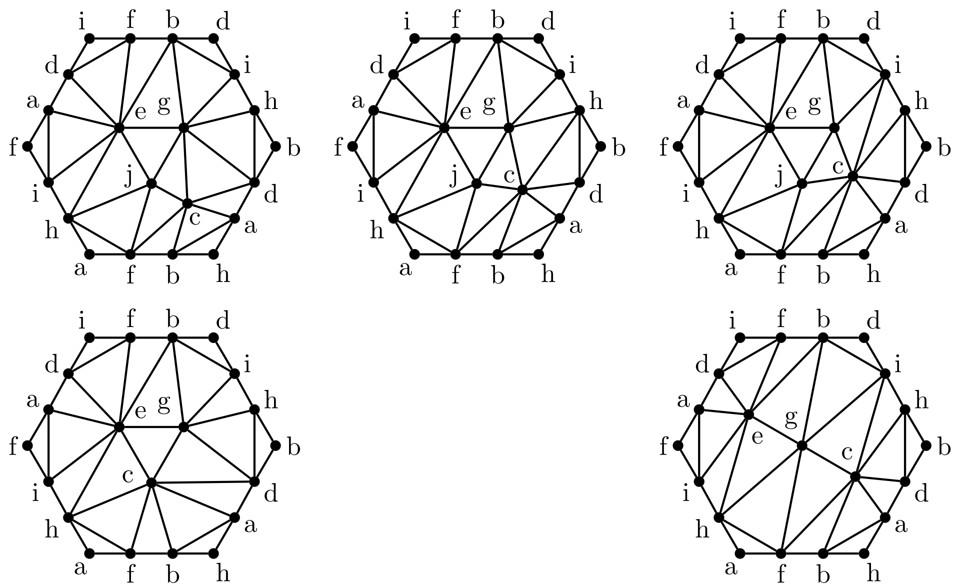


FIGURE 28. Portage from $P(N_4, 9, 3, 1)$ to $P(N_4, 9, 1, 15)$

11. N_5

We have now examined all of the surfaces for which all of the pseudo-minimal triangulations have been determined. We will now examine the results found by the algorithm described in Section 2.

Two frozen pseudo-minimal triangulations were the only ones found for N_5. Each fills its own lake. The two pseudo-minimal triangulations and the connecting portage are shown in Figure 29. Since any pseudo-minimal triangulations with 9 vertices on N_5 must be complete we have from [2] that these are the only pseudo-minimal triangulations with 9 vertices. There are no pseudo-minimal triangulations with 10 vertices and we conjecture that there are no pseudo-minimal triangulations with greater than 10 vertices. If this is so then $N(N_5) = L_{max}(N_5)+1 = V_{min}(N_5)+1 = 10$.

12. S_3

There were four lakes found for S_3. Two of the lakes, $L(S_3, 10, 3)$ and $L(S_3, 10, 4)$, each consists of one isolated pseudo-minimal triangulation. One of these isolated pseudo-minimal triangulation, $P(S_3, 10, 3, 1)$, is frozen and is shown in Figure 30. It is $K_{10} - K_3$ and its rotation is a relabeled version of rotation (2.8) in [17]. By Ringel's construction, vertices h, i, and j are separated by two other vertices in each line of the rotation in which they appear. Thus no pair of the three vertices are opposite each other and $P(S_3, 10, 3, 1)$ is frozen. The construction for the orientable case 10 of [17] provides a triangulation of $K_{12s+10} - K_3$ in $S_{(4s+3)(3s+1)}$ for every $s \geq 1$ with a rotation in which the vertices of K_3 are separated by at least two other vertices in each line of the rotation in which they appear. Korzhik and Voss [6] provide more than one such construction for $s \geq 2$. So for $s \geq 2$ we have $N(S_{(4s+3)(3s+1)}) \geq V_{min}(S_{(4s+3)(3s+1)}) + 1 = 12s + 11$.

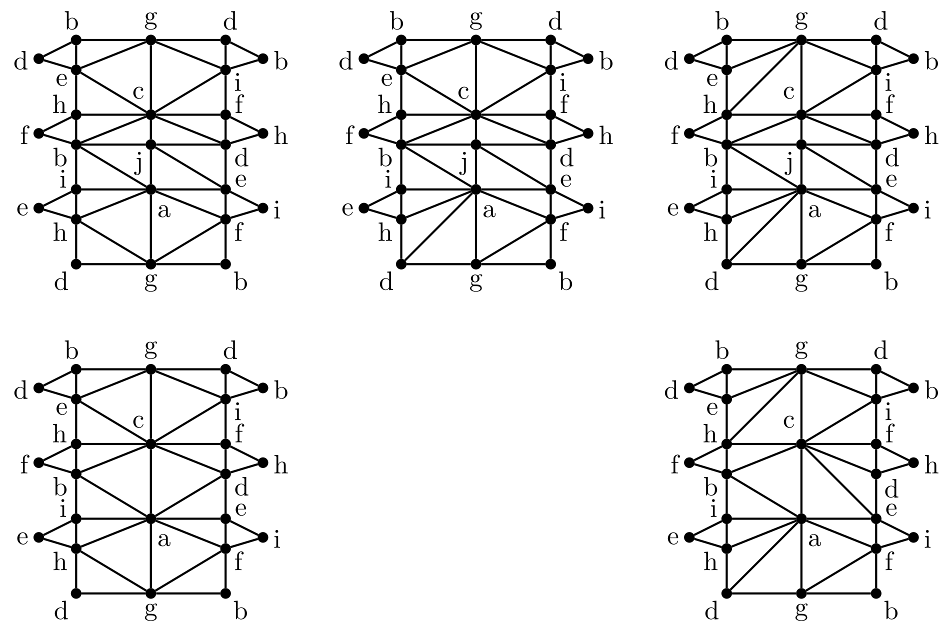
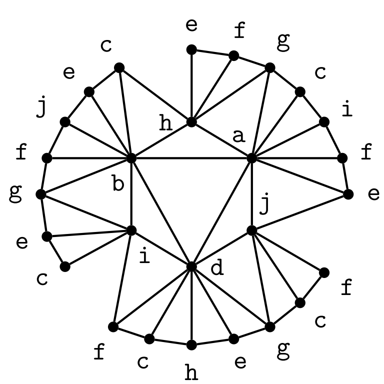


FIGURE 29. Portage from $P(N_5, 9, 1, 1)$ to $P(N_5, 9, 2, 1)$



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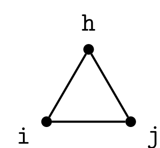
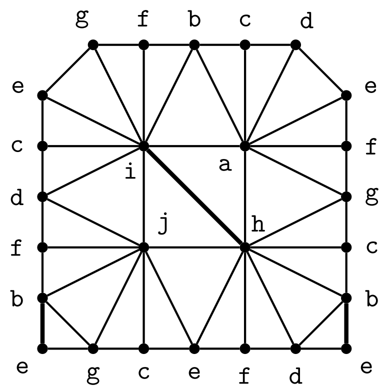


FIGURE 30. $P(S_3, 10, 3, 1)$

The other isolated pseudo-minimal triangulation of S_3, $P(S_3, 10, 4, 1)$, is shown in Figure 31. It is nonfrozen as its two equivalent flippable edges are self-flippable. That the two edges, hi and be, are equivalent can be seen from the generator for the automorphism group of the triangulation. If we flip hi to get aj and relabel the vertices with the permutation, $(ai)(cf)(dg)(hj)$, we obtain a reflection of the original drawing. $P(S_3, 10, 4, 1)$ is the smallest nonfrozen isolated pseudo-minimal triangulation of an orientable surface.

Figure 32 shows a third pseudo-minimal triangulation of S_3. It is not isolated. We have shown three pseudo-minimal triangulations of S_3 each from a different lake. We will not show that there are at least four lakes. The portage graph of S_3 is K_4. The minimal spanning tree is P_4 with each edge having a weight

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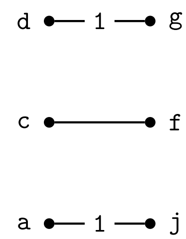
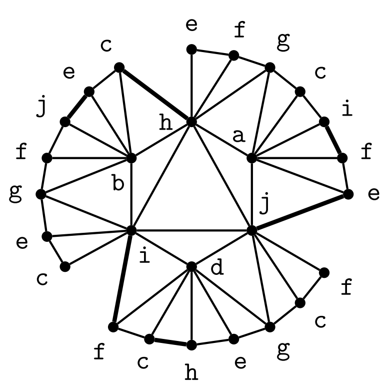


FIGURE 31. $P(S_3, 10, 4, 1)$

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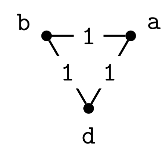


FIGURE 32. $P(S_3, 10, 2, 1)$

of 3. We conjecture that these 20 triangulations which are all of the minimal triangulations are also all of the pseudo-minimal triangulations. If this is so then $N(S_3) = L_{max}(S_3) + 1 = V_{min}(S_3) + 1 = 11$.

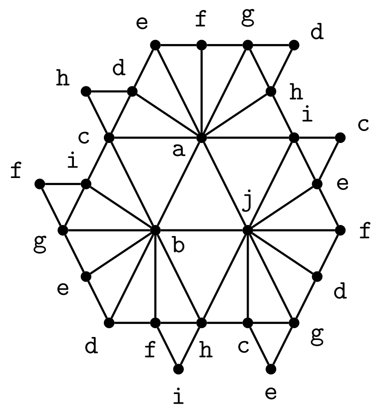
13. N_6

For N_6 there were 1022 pseudo-minimal triangulations found in 22 lakes. As with S_3 two of the pseudo-minimal triangulations are isolated. One is frozen and is shown in Figure 33 while the other is not frozen and is shown in Figure 34. $P(N_6, 10, 22, 1)$ is the smallest nonfrozen isolated pseudo-minimal triangulation of a nonorientable surface.

The portage graph of N_6 is connected. All edges of the minimal spanning tree have weight of 3 except for one edge. All portages from one of the lakes have weight 4 or more. We conjecture that there are 1022 pseudo-minimal triangulations in 22 lakes. If this is so then $N(N_6) = L_{max}(N_6) + 1 = V_{min}(N_6) + 1 = 11$.

14. N_7

The 14 complete triangulations of N_7 [2] with 10 vertices were found. Examples of pseudo-minimal triangulations with 11 vertices were also found. These 20



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(efd)(cih)(jba)

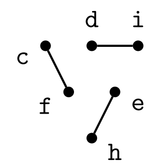
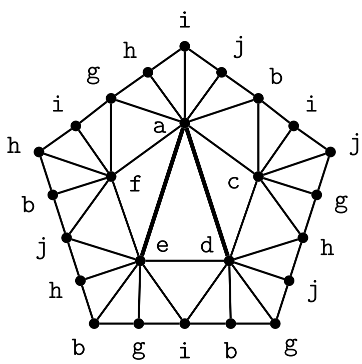


FIGURE 33. $P(N_6, 10, 21, 1)$



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j aicgdhefb

(cf)(de)(gb)(hj)

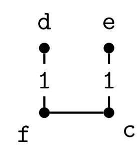
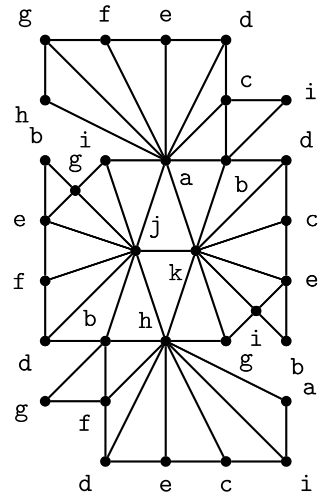


FIGURE 34. $P(N_6, 10, 22, 1)$

pseudo-minimal triangulations are not minimal. They are the smallest nonminimal pseudo-minimal triangulations found for any surface. One isolated nonminimal pseudo-minimal triangulations was found. It is frozen and is shown in Figure 35. There are also nonisolated nonminimal pseudo-minimal triangulations. To show there existence we display in Figures 36 through 38 the three pseudo-minimal triangulations from $L(N_7, 11, 4)$ which are equivalent under diagonal flips to each other but are not equivalent under diagonal flips to any other triangulations.

The portage graph is not connected because N_7 has lakes at two elevations. The portage graph consists of two connected components. The portage graph restricted to lakes of elevation 10 has a minimal spanning tree with weights ranging from 3 to 12. The portage graph restricted to lakes of elevation 11 has a minimal spanning tree with weights ranging from 10 to 13. It is possible to transform any pseudo-minimal triangulation with 11 vertices into a pseudo-minimal triangulation with 10 vertices by splitting a vertex, a sequence of diagonal flips, and then two edge contractions.

We conjecture that the pseudo-minimal triangulations which were found are all of the pseudo-minimal triangulations of N_7. If this is so then $N(N_7) = L_{max}(N_7)+1 = V_{min}(N_7) + 2 = 12$.



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(cf)(ig)(kj)(ah)
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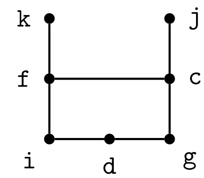
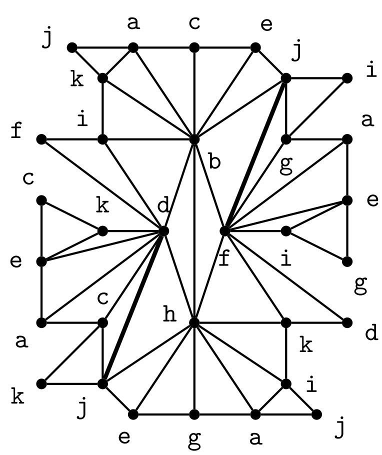


FIGURE 35. $P(N_7, 11, 5, 1)$



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(cg)(bh)(fd)(ik)
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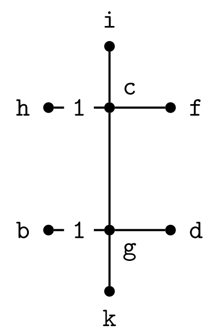
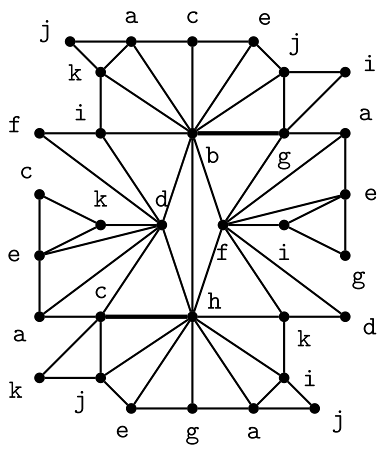


FIGURE 36. $P(N_7, 11, 4, 1)$

15. S_4

For S_4 there were 821 pseudo-minimal triangulations with 11 vertices found in 12 lakes. The 3 isolated pseudo-minimal triangulations with 11 vertices are frozen. The minimal spanning tree for these lakes exists and the weight of all edges of this tree is 3. There were 2 frozen triangulations with 12 vertices found. The portage graph restricted to lakes of elevation 12 is a single edge with weight 3. It is possible to transform either pseudo-minimal triangulation with 12 vertices into a pseudo-minimal triangulation with 11 vertices by splitting a vertex, a sequence of diagonal flips, and then two edge contractions.

We conjecture that the pseudo-minimal triangulations which were found are all of the pseudo-minimal triangulations of S_4. If this is so then $N(S_4) = L_{max}(S_4)+1 = V_{min}(S_4) + 2 = 13$.

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j aigbehck
k ajcedfhib

(cg)(df)(ki)(bh)
```

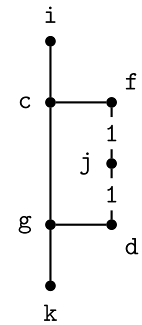
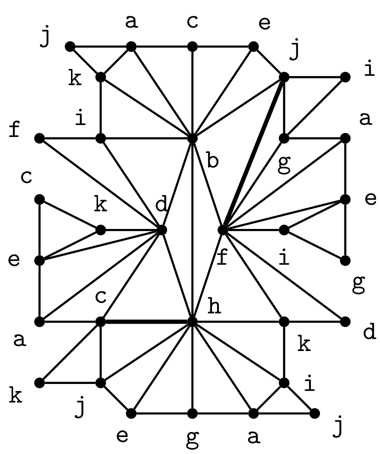


FIGURE 37. $P(N_7, 11, 4, 2)$

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j aigfbehck
k ajcedfhib
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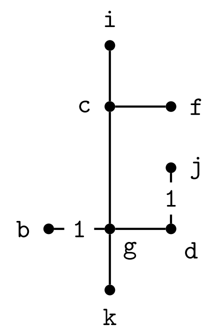

FIGURE 38. $P(N_7, 11, 4, 3)$

16. OTHER SURFACES

As the genus of the surface increases it becomes more difficult to find a set of pseudo-minimal triangulations which appear to be all for that surface. At least two frozen pseudo-minimal triangulations were found for all surfaces with $11 <= V_{min} <= 17$. We conjecture that $N(S) = V_{min}(S)$ only if S is one of the surfaces: S_0, S_1, S_2, N_1, N_2, or N_3.

17. LABELED TRIANGULATIONS

If a triangulation has n vertices we obtain a *labeled triangulation* by assigning a unique label to each vertex. We will use letters for the labels. Two labeled triangulations T and T' of a surface are *equivalent* if there is a isomorphism h with $h(T) = T'$ which preserves the labels. That is, if a is a vertex of T then the label assigned to a is the label assigned to the vertex $h(a)$ of T'.

For a triangulation T with n vertices all of the labeled triangulations obtained from T form a symmetric group on n objects. The subgroup of labeled triangulations obtained from T which are equivalent is the automorphism group of T.

Two labeled triangulations are *equivalent under diagonal flips* if one is equivalent as a labeled triangulation to a labeled triangulation obtained from the other by a sequence of diagonal flips.

Suppose that ab is a flippable edge of a labeled triangulation T, that the labeled triangulation T' is obtained by flipping ab in T and replacing it with the edge cd in T', and that h is an automorphism of T. Then the two edges ab and $h(a)h(b)$ of T are equivalent. By flipping the edge $h(a)h(b)$ of T we obtain a labeled triangulation T'' which is equivalent as an unlabeled triangulation to T'. h maps the labels of T' onto the labels of T''. If we start with T', flip cd to obtain T, and then flip $h(a)h(b)$ to obtain T'' we see that the labeled triangulations T' and T'' are equivalent under diagonal flips. So the automorphism group of T can be used to relabel the vertices of T' to obtain other labeled triangulations which are equivalent under diagonal flips to T'. In general, the vertices of a labeled triangulation can be relabeled to obtain other labeled triangulations which are equivalent under diagonal flips by using the automorphism group of any triangulation which is equivalent under diagonal flips. If we consider a set of labeled triangulations which are equivalent under diagonal flips and show that the generators of the automorphism groups of these triangulations generate the symmetric group on the labels then all the possible labelings of these triangulations are equivalent under diagonal flips.

Let $N_L(S)$ be the minimum value such that two labeled triangulations T and T' are equivalent under diagonal flips if the number of vertices in T and the number of vertices in T' are equal and at least $N_L(S)$. Negami [12] has shown that $N(S) \leq N_L(S) \leq N(S)+1$ and that if $V_{min}(S) < N(S)$ then $N(S) = N_L(S)$.

We will examine $N_L(S)$ when $V_{min}(S) = N(S)$. In this case, from Theorem 1 S has exactly one lake. The surfaces which we have examined which do or might meet this criteria are S_0, N_1, S_1, N_2, N_3, and S_2.

Negami [12] has shown that $N_L(S_0) = N(S_0) = 4$, $N_L(N_1) = N(N_1)+1 = 7$, and $N_L(S_1) = N(S_1)+1 = 8$.

We will show that $N_L(N_2) = N(N_2) = 8$. From Figure 8 a generator of the automorphism group of Kh2 is $h_1 = (ac)(dh)(eg)$ and from Figure 10 two generators of the automorphism group of Kh3 are $h_2 = (bc)(ef)(gh)$ and $h_3 = (ahdg)(be)(cf)$. $h_2h_3h_1 = (ade)(bf)(cgh)$, $(h_2h_3h_1)^3 = (bf)$, $h_2h_3h_1h_2h_1 = (ahbgecdf)$. Thus combining the three permutations h_1, h_2, and h_3 we can produce a 2-cycle and an n-cycle and so these three permutations generate the symmetric group on the 8 labels of the minimal triangulations of N_2. Therefore all labeled triangulations which are obtained by labeling the 6 minimal triangulations of N_2 are equivalent under diagonal flips so $N_L(N_2) = N(N_2) = 8$. In particular, any relabeling of the vertices of Kh2 can be obtained by some sequence of the operations: redraw Kh2 maintaining the structure shown in Figure 8 which permutes the labels with h_1; flip edges to obtain Kh3 from Kh2; redraw Kh3 maintaining the structure shown in Figure 10 which permutes the labels with some combination of h_2 and h_3; and flip edges to obtain Kh2 from Kh3.

We will show that $N_L(N_3) = N(N_3) = 9$. Figure 12 can be obtained from Figure 11 by flipping bi, ah, ci, and df. From Figure 11 the generator of the automorphism group of $P(N_3, 9, 1, 1)$ is $h_1 = (abd)(cfe)(ghi)$ and from Figure 12 the generator of the automorphism group of $P(N_3, 9, 1, 2)$ is $h_2 = (ag)(be)(dh)$. $h_1h_2 = (aecfbhi)(dg)$ and $(h_1h_2)^7 = (dg)$, a 2-cycle. $h_1h_1h_2 = (ah)(bgidefc)$ and $(h_1h_1h_2)^2(h_1h_2)^4 = (abfeihcgd)$, an n-cycle. So h_1 and h_2 generate the symmetric group on the 9 labels of $P(N_3, 9, 1, 1)$ and thus any member of $L(N_3, 9, 1)$. If $L(N_3, 9, 1)$ is the only lake for N_3 then we have just shown that $N_L(N_3) = 9$ which would also be the value for $N(N_3)$. If there is more than one lake for N_3 then $V_{min}(N_3) < N(N_3)$ and so $N_L(N_3) = N(N_3)$.

In a similar way $N_L(S_2) = N(S_2) = 10$. From Figures 17 through 19 the respective generators of the automorphism groups of $P(S_2, 10, 1, 1)$, $P(S_2, 10, 1, 2)$, and $P(S_2, 10, 1, 3)$ are $h_1 = (ai)(bj)(ce)(dh)(fg)$, $h_2 = (ai)(bh)(dj)$, and $h_3 = (ab)(ci)(fh)$. $h_3h_1h_3h_2h_1 = (ahfbeicdgj)$, a 10-cycle. $h_3h_2h_3h_1h_3h_2h_1 = (afeid)(bch)(gj)$ so $(h_3h_2h_3h_1h_3h_2h_1)^{15} = (gj)$, a 2-cycle. Thus, h_1, h_2, and h_3 generate the symmetric group on the 10 labels of $P(S_2, 10, 1, 1)$.

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